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Abstract. We establish a new lower bound on the first nonzero eigenvalue $\lambda_1(\theta)$ of the sublaplacian Δ_b on a compact strictly pseudoconvex CR manifold M carrying a contact form θ whose Tanaka-Webster connection has pseudohermitian Ricci curvature bounded from below.

1. Introduction and statement of main result

By a classical result of A. Lichnerowicz (cf. Theorem D.I.1 in [8], p. 179) and M. Obata (cf. [34]) for any m-dimensional compact Riemannian manifold (M, g) with $\mathrm{Ric}_g \geq k\, g$ the first nonzero eigenvalue $\lambda_1(g)$ of the Laplace-Beltrami operator Δ_g satisfies the estimate

$$\lambda_1(g) \geq mk/(m-1) \tag{1}$$

with equality if and only if M is isometric to the unit sphere $S^m \subset \mathbb{R}^{m+1}$. The main ingredient in the proof of (1) is the Bochner-Lichnerowicz formula (cf. e.g. (G.IV.5) in [8], p. 131)

$$-\frac{1}{2}\Delta_g\left(\|du\|^2\right) = \|\mathrm{Hess}(u)\|^2 - g\left(Du\,,\ D\Delta_g u\right) + \mathrm{Ric}_g(Du, Du) \tag{2}$$

for any $u \in C^\infty(M, \mathbb{R})$. The great fascination exerted by the Lichnerowicz-Obata theorem on the mathematical community in the last fifty years prompted the many attempts to extend (2) and (1) to other geometric contexts e.g. to Riemannian foliation theory (cf. S-D. Jung & K-R. Lee & K. Richardson, [27], J. Lee & K. Richardson, [31], H-K. Pak & J-H. Park, [36]), to CR and pseudohermitian geometry (cf. E. Barletta & S. Dragomir, [3], E. Barletta, [4], S-C. Chang & H-L. Chiu, [10], H-L. Chiu, [11], A. Greenleaf, [23], S. Ivanov & D. Vassilev, [24], S-Y. Li & H-S. Luk, [32]) and to sub-Riemannian geometry (cf. F. Baudoin & N. Garofalo, [7]). The present paper is devoted to a version of the estimate (1) occurring in CR geometry. Given a compact strictly pseudoconvex CR manifold $(M, T_{1,0}(M))$ endowed with a positively oriented contact form θ, the pseudohermitian manifold

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(M, θ) carries a natural second order, positive, formally self-adjoint operator Δ_b (the *sublaplacian* of (M, θ)), formally similar to the Laplacian in Riemannian geometry, yet only degenerate elliptic (in the sense of J-M. Bony, [9]). However Δ_b is hypoelliptic and (by a result of A. Menikoff & J. Sjöstrand, [33]) it has a discrete spectrum

$$\sigma(\Delta_b) = \{\lambda_\nu(\theta) : \nu \in \mathbb{Z}, \quad \nu \geq 0\}, \quad \lim_{\nu \to \infty} \lambda_\nu = +\infty,$$

$$\lambda_0(\theta) = 0, \quad \lambda_\nu(\theta) \leq \lambda_{\nu+1}(\theta), \quad \nu \geq 1.$$

On the other hand, by a result of N.Tanaka, [37], and S.M. Webster, [38], (M, θ) carries a natural linear connection ∇ (the *Tanaka-Webster connection* of (M, θ), cf. also [12], p. 25) whose Ricci tensor field is formally similar to Ricci curvature in Riemannian geometry. It is then a natural problem to look for a lower bound on $\lambda_1(\theta)$ whenever Ric_∇ is bounded from below. By strict pseudoconvexity (M, θ) also carries a natural Riemannian metric g_θ (the *Webster metric* of (M, θ), cf. [12], p. 9) whose associated Riemannian volume form is (up a multiplicative constant depending only on the CR dimension n) $\Psi_\theta = \theta \wedge (d\theta)^n$. Let $\mathrm{div} : \mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$ be the divergence operator associated to the volume form Ψ_θ. Then the sublaplacian may be written in divergence form as $\Delta_b u = -\mathrm{div}(\nabla^H u)$ where $\nabla^H u$ (the *horizontal gradient*) is the projection of the gradient ∇u with respect to g_θ, on the Levi, or maximally complex, distribution $H(M) = \mathrm{Re}\,\{T_{1,0}(M) \oplus T_{0,1}(M)\}$. Consequently the horizontal gradient $\nabla^H u$ is the pseudohermitian analog to the gradient Du in Riemannian geometry. The first step is then to produce a pseudohermitian version of (2) i.e. compute $\Delta_b(\|\nabla^H u\|^2)$ (for an arbitrary eigenfunction u of Δ_b) in terms of the pseudohermitian Hessian $\nabla^2 u$ and the Ricci curvature Ric_∇ of the Tanaka-Webster connection. The first to realize the difficulties in producing a pseudohermitian analog to (2) was A. Greenleaf, [23]. Indeed his Bochner-Lichnerowicz type formula

$$(3) \qquad \frac{1}{2}\Delta_b\left(\|\nabla^{1,0}u\|^2\right) = \sum_{\alpha,\beta}\left(u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta} + u_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}}\right) + 2i\sum_{\alpha}\left(u_{\overline{\alpha}}u_{0\alpha} - u_\alpha u_{0\overline{\alpha}}\right) +$$

$$+ \sum_{\alpha,\beta} R_{\alpha\overline{\beta}}u_{\overline{\alpha}}u_\beta + in\sum_{\alpha,\beta}\left(A_{\alpha\beta}u_{\overline{\alpha}}u_{\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}u_\alpha u_\beta\right) + \frac{1}{2}\sum_\alpha\{u_{\overline{\alpha}}(\Delta_b u)_\alpha + u_\alpha(\Delta_b u)_{\overline{\alpha}}\}$$

involves the torsion terms $A_{\alpha\beta}$ (possessing no Riemannian counterpart). Here $\nabla^{1,0}u = \sum_\alpha u_{\overline{\alpha}} T_\alpha$ (notations and conventions as used in (3) are explained in § 2). However the attempt to confine oneself to the class of Sasakian manifolds (M, g_θ) (as in [4], since Sasakian metrics g_θ have vanishing pseudohermitian torsion i.e. $A_{\alpha\beta} = 0$) isn't successful either: while torsion terms may actually be controlled (when exploiting (3) integrated over M) by the L^2 norm of $\nabla^H u$, the main technical difficulties really arise from the occurrence of terms $\sum_\alpha (u_{\overline{\alpha}}u_{0\alpha} - u_\alpha u_{0\overline{\alpha}})$ containing covariant derivatives of $\nabla^H u$ in the "bad" real direction T transverse to $H(M)$ (the *Reeb vector* field of (M, θ)).

The novelty brought by the present paper is to establish first a version of Bochner-Lichnerowicz formula for a natural Lorentzian metric F_θ (the *Fefferman metric* of (M, θ), cf. [29], [21]) on the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$. Fefferman metric F_θ was discovered by C. Fefferman, [20], in connection with the study of boundary behavior of the Bergman kernel of a strictly pseudoconvex domain in $\mathbb{C}^n$. An array of problems of major interest in CR geometry e.g. the CR Yamabe problem, [25], the study of subelliptic harmonic maps, [26], and Yang-Mills fields on CR manifolds, [6], are closely tied to the geometry of the Lorentzian manifold $(C(M), F_\theta)$. Indeed the aforementioned problems are projections on M via $\pi : C(M) \to M$ of Lorentzian analogs to the corresponding Riemannian problems, as prompted by J.M. Lee's discovery (cf. [29]) that $\pi_* \Box = \Delta_b$, where $\Box$ is the Laplace-Beltrami operator of F_θ (the wave operator on $(C(M), F_\theta)$). For instance any S^1-invariant harmonic map $\Phi : (C(M), F_\theta) \to N$ into a Riemannian manifold N projects on a subelliptic harmonic map $\phi : M \to N$ (in the sense of [26] and [6]). The arguments in [8] carry over in a straightforward manner (cf. our § 3) to Lorenzian geometry and give (cf. (21) below)

$$-\frac{1}{2} \Box \left(F_\theta(Df, Df)\right) = F_\theta^*\left(D^2 f\, ,\, D^2 f\right) - (Df)(\Box f) + \mathrm{Ric}_D(Df\, ,\, Df) \tag{4}$$

and the corresponding integral formula (22). The projection on M of (4) then leads to another analog (similar to A. Greenleaf's formula (3)) to Bochner-Lichnerowicz formula and then to a new lower bound on $\lambda_1(\theta)$. Precisely we may state

Theorem 1. *Let M be a compact, strictly pseudoconvex, CR manifold of CR dimension n. Let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on M and Δ_b the corresponding sublaplacian. Let* Ric_∇ *be the Ricci tensor of the Tanaka-Webster connection ∇ of (M, θ) and $\lambda_1(\theta) \in \sigma(\Delta_b)$ the first nonzero eigenvalue of Δ_b. If*

$$\mathrm{Ric}_\nabla(X, X) \geq k\, G_\theta(X, X) \tag{5}$$

for some constant $k > 0$ and any $X \in H(M)$ then

$$\lambda_1(\theta) \geq \frac{2n}{(n+2)(n+3)} \left\{ (n+3)k - (11n+19)\tau_0 - \frac{\rho_0}{2(n+1)} \right\} \tag{6}$$

with $\tau_0 = \sup_{x \in M} \|\tau\|_x$ and $\rho_0 = \sup_{x \in M} \rho(x) \geq nk$, where τ and ρ are respectively the pseudohermitian torsion and scalar curvature of (M, θ).

The lower bound (6) is nontrivial only for k sufficiently large (i.e. k must satisfy (89) in § 6). Let (M, g_θ) be a Sasakian manifold (equivalently $\tau = 0$, cf. e.g. [12]). Then under the same assumption (i.e. (5) in Theorem 1) A. Greenleaf established the estimate (cf. [23])

$$\lambda_1(\theta) \geq \frac{nk}{n+1} . \tag{7}$$

Lower bound (6) is sharper that (7) when

$$k > \frac{\rho_0}{n(n+3)} . \tag{8}$$

If for instance $M = S^{2n+1}$ is the standard sphere in $\mathbb{C}^{n+1}$, endowed with the canonical contact form $\theta = (i/2)\left(\overline{\partial} - \partial\right)|z|^2$, then $\rho_0 = 2n(n+1)$ and $k = 2(n+1)$ hence (8) holds (and (6) is sharper than (7)).

The essentials of CR and pseudohermitian geometry are recalled in § 2 (by following mainly [12]). The projection of (4) on M gives

$$\begin{gathered} -\frac{1}{2}\Delta_b\left(\|\nabla^H u\|^2\right) = \left\|\Pi_H \nabla^2 u\right\|^2 - (\nabla^H u)(\Delta_b u) + \\ +4(J\nabla^H u)(u_0) - \frac{3(n+1)}{n+2} A(\nabla^H u\,,\, J\nabla^H u) + \\ +\frac{n+3}{n+2}\operatorname{Ric}_\nabla\left(\nabla^H u\,,\, \nabla^H u\right) - \frac{\rho}{2(n+1)(n+2)}\|\nabla^H u\|^2 \end{gathered} \tag{9}$$

(the *pseudohermitian Bochner-Lichnerowicz* formula, cf. (79) in § 5) and the corresponding integral formula (80). The main technical difficulty in the derivation of (9) is to compute the Ricci curvature Ric_D of the Lorentzian manifold $(C(M), F_\theta)$. This is performed by relating the Levi-Civita connection D of $(C(M), F_\theta)$ to the Tanaka-Webster connection ∇ of (M, θ) (cf. (23)-(27) in § 4, a result got in [6]) and adapting to $S^1 \to C(M) \to M$ a technique originating in the theory of Riemannian submersions (cf. [35]) and shown to work in spite of the fact that $\pi : (C(M), F_\theta) \to (M, g_\theta)$ isn't a semi-Riemannian submersion (fibres of π are degenerate). The relationship among D and ∇ may then be exploited to compute the full curvature tensor R^D. Only its trace Ric_D is evaluated in [29] and the formula there appears as too involved to be of practical use. Our result (cf. (50)-(54) in Lemma 3 below) is simple, elegant and local frame free. This springs from the decompositions $T(C(M)) = \operatorname{Ker}(\sigma) \oplus \mathbb{R}S$ and $\operatorname{Ker}(\sigma) = H(M)^\uparrow \oplus \mathbb{R}T^\uparrow$, themselves relying on the discovery (due to C.R. Graham, [21]) that $\sigma \in \Omega^1(C(M))$ (given by (20) below) is a connection 1-form in the principal circle bundle $S^1 \to C(M) \to M$. As a byproduct of Lemma 3 one reobtains the result by J.M. Lee, [29], that none of the Fefferman metrics $\{F_\theta \in \operatorname{Lor}(C(M)) : \theta \in \mathcal{P}_+\}$ is Einstein. Integration of (9) over M produces (by (78)) terms $\|u_0\|_{L^2}$ where $u_0 \equiv T(u)$ and u is an arbitrary eigenfunction of Δ_b, corresponding to a fixed eigenvalue $\lambda \in \sigma(\Delta_b)$. The L^2 norm of the (restriction to the Levi distribution $H(M)$ of the) pseudohermitian Hessian $\Pi_H \nabla^2 u$ is estimated by using (82) (a result got in [4]). Torsion terms and Ricci curvature terms are respectively estimated by (87) and as a consequence of the assumption (5) in Theorem 1 (together with (86)). Finally to control $\|u_0\|_{L^2}$ one exploits a fundamental result got in [10], and referred hereafter as the *Chang-Chiu inequality* (cf. (91) in Appendix A).

Spectral geometry (spectrae of Laplace-Beltrami and Schrödinger operators) on compact Riemannian manifolds has been intensely investigated over the last twenty-five years by A. El Soufi & S. Ilias, [13], A. El Soufi & S. Ilias & A. Ros, [14], A. El Soufi & B. Colbois, [15], A. El Soufi & B. Colbois & E. Dryden, [16], A. El Soufi & N. Moukadem, [17], A. El Soufi & H. Giacomini & M. Jazar, [18], and A. El Soufi & R. Kiwan, [19]. A program aiming to recovering the quoted works in the realm of CR and pseudohermitian geometry was recently started by A. Aribi & A. El Soufi, [1]. As part of that program A. Aribi & S. Dragomir & A. El Soufi studied (cf. [2]) the dependence of spectrae of sublaplacians on the given contact form. The present work is another step on this path (studying spectrae of compact strictly pseudoconvex CR manifolds).

The paper is organized as follows. In § 2 we recall the needed elements of calculus on a pseudohermitian manifold (including the curvature theory for the Tanaka-Webster conection, cf. [37], [38] and [12]). The Lorentzian analog (4) to the Bochner-Lichnerowicz formula (2) is derived in § 3. The technicalities on curvature theory (needed to project (4) on M) are dealt with in § 4. In § 5 we relate the Lorentzian Hessian $D^2(u \circ \pi)$ to the pseudohermitian Hessian $\nabla^2 u$ and derive the pseudohermitian Bochner-Lichnerowicz formula (9). The lower bound (1) is proved in § 6. Appendix A contains a proof of the Chang-Chiu inequality.

2. A reminder of CR geometry

For all definitions and basic conventions in CR and pseudohermitian geometry we rely on [12]. Let $(M, T_{1,0}(M))$ be an orientable CR manifold, of CR dimension n, and let $\overline{\partial}_b$ be the tangential Cauchy-Riemann operator. A CR function is a C^1 solution to the tangential C-R equations $\overline{\partial}_b f = 0$. Let $H(M)$ be the maximally complex, or Levi, distribution on M and let J be its complex structure. Let $\mathcal{P}$ be the space of all pseudohermitian structures on M. For each $\theta \in \mathcal{P}$ the Levi form is $G_\theta(X, Y) = (d\theta)(X, JY)$ for every $X, Y \in H(M)$. If M is nondegenerate then each $\theta \in \mathcal{P}$ is a contact form i.e. $\Psi_\theta = \theta \wedge (d\theta)^n$ is a volume form on M. When M is strictly pseudoconvex we denote by $\mathcal{P}_+$ the set of positively oriented contact forms i.e. all $\theta \in \mathcal{P}$ such that G_θ is positive definite. If M is nondegenerate then for each contact form $\theta \in \mathcal{P}$ there is a unique nowhere zero, globally defined, vector field $T \in \mathfrak{X}(M)$ (the Reeb vector field of (M, θ)) such that $\theta(T) = 1$

and $T \rfloor d\theta = 0$. By taking into account the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ one may extend the Levi form G_θ to a semi-Riemannian metric g_θ (the Webster metric of (M, θ)) given by $g_\theta(X, Y) = G_\theta(X, Y)$, $g_\theta(X, T) = 0$ and $g_\theta(T, T) = 1$, for every $X, Y \in H(M)$. By a fundamental result of N. Tanaka, [37], and S. Webster, [38], for each contact form $\theta \in \mathcal{P}$ there is a unique linear connection ∇ (the Tanaka-Webster connection of (M, θ)) such that i) $H(M)$ is parallel with respect to ∇, ii) $\nabla g_\theta = 0$, $\nabla J = 0$, and iii) the torsion tensor field T_∇ is pure i.e. $T_\nabla(Z, W) = 0$, $T_\nabla(Z, \overline{W}) = 2iG_\theta(Z, \overline{W})T$ and $\tau \circ J + J \circ \tau = 0$, for any $Z, W \in T_{1,0}(M)$. For all local calculations we consider a local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$, defined on the open set U, and set

$$g_{\alpha\overline{\beta}} = G_\theta(T_\alpha\,,\, T_{\overline{\beta}}), \quad T_{\overline{\alpha}} = \overline{T}_\alpha\,, \quad \nabla T_B = \omega_B{}^A T_A\,,$$

$$\omega_B{}^A = \Gamma^A_{CB}\theta^C\,, \quad \tau(T_\alpha) = A^{\overline{\beta}}_\alpha T_{\overline{\beta}}\,, \quad A_{\alpha\beta} = g_{\alpha\overline{\gamma}} A^{\overline{\gamma}}_\beta\,,$$

$$\alpha, \beta, \gamma, \cdots \in \{1, \cdots, n\}, \quad A, B, C, \cdots \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}.$$

Here $\{\theta^\alpha : 1 \le \alpha \le n\}$ is the adpated coframe determined by $\theta^\alpha(T_\beta) = \delta^\alpha_\beta$, $\theta^\alpha(T_{\overline{\beta}}) = 0$ and $\theta^\alpha(T) = 0$. Then (cf. e.g. (1.62) and (1.64) in [12], p. 39-40)

$$d\theta = 2i g_{\alpha\overline{\beta}}\,\theta^\alpha \wedge \theta^{\overline{\beta}}\,, \quad d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + \theta \wedge \tau^\alpha\,, \quad A_{\alpha\beta} = A_{\beta\alpha}\,, \tag{10}$$

where $\tau^\alpha \equiv A^\alpha_{\overline{\beta}}\theta^{\overline{\beta}}$ and $A^\alpha_{\overline{\beta}} = \overline{A^{\overline{\alpha}}_\beta}$. Therefore, if we set $A(X, Y) = g_\theta(\tau X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ then A is symmetric. Let R^∇ be the curvature tensor field of the Tanaka-Webster connection ∇. As to the local components of R^∇ we adopt the convention $R^\nabla(T_B, T_C)T_A = R_A{}^D{}_{BC} T_D$ (cf. [12], p. 50). The Ricci tensor of ∇ is $\mathrm{Ric}_\nabla(Y, Z) = \mathrm{trace}\left\{X \in T(M) \longmapsto R^\nabla(X, Z)Y\right\}$ for any $Y, Z \in T(M)$. Locally we set $R_{AB} = \mathrm{Ric}_\nabla(T_A, T_B)$. The pseudohermitian Ricci tensor is then $R_{\lambda\overline{\mu}}$. By a result of S. Webster, [38] (to whom the notion is due) $R_{\lambda\overline{\mu}} = R_\lambda{}^\alpha{}_{\alpha\overline{\mu}}$. The pseudohermitian scalar curvature is $\rho = g^{\lambda\overline{\mu}} R_{\lambda\overline{\mu}}$ where $\left[g^{\alpha\overline{\beta}}\right] = \left[g_{\alpha\overline{\beta}}\right]^{-1}$. Let us set

$$\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta\,, \quad \Omega_\alpha{}^\beta = \Pi_\alpha{}^\beta - 2i\theta_\alpha \wedge \tau^\beta + 2i\tau_\alpha \wedge \theta^\beta\,,$$

where $\theta_\alpha = g_{\alpha\overline{\beta}}\theta^{\overline{\beta}}$, $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$, $\tau_\alpha = g_{\alpha\overline{\beta}}\tau^{\overline{\beta}}$ and $\tau^{\overline{\beta}} = A^{\overline{\beta}}_\alpha \theta^\alpha$. By a result of S.M. Webster, [38] (cf. also Theorem 1.7 in [12], p. 55)

$$\Omega_\alpha{}^\beta = R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\,\theta^\lambda \wedge \theta^{\overline{\mu}} + W^\beta_{\alpha\lambda}\theta^\lambda \wedge \theta - W^\beta_{\alpha\overline{\lambda}}\theta^{\overline{\lambda}} \wedge \theta \tag{11}$$

where $W^\beta_{\alpha\overline{\mu}} = g^{\beta\overline{\sigma}}\,\nabla_\alpha A_{\overline{\mu}\overline{\sigma}}$ and $W^\beta_{\alpha\lambda} = g^{\beta\overline{\sigma}}\nabla_{\overline{\sigma}} A_{\alpha\lambda}$. Given $u \in C^\infty(M, \mathbb{R})$ the pseudohermitian Hessian is $(\nabla^2 u)(X, Y) = (\nabla_X du)Y$ for any $X, Y \in \mathfrak{X}(M)$. Locally we set $\nabla_A u_B = (\nabla^2 u)(T_A, T_B)$. The pseudohermitian Hessian is not symmetric. Rather one has the commutation formulae

$$\nabla_\alpha u_\beta = \nabla_\beta u_\alpha\,, \quad \nabla_\alpha u_{\overline{\beta}} = \nabla_{\overline{\beta}} u_\alpha - 2i g_{\alpha\overline{\beta}} u_0\,, \quad u_0 \equiv T(u), \tag{12}$$

$$\nabla_0 u_\beta = \nabla_\beta u_0 - u_{\overline{\alpha}} A^{\overline{\alpha}}_\beta \,. \tag{13}$$

The 3rd order covariant derivative of u is $(\nabla^3 u)(X,Y,Z) = (\nabla_X H_u)(Y,Z) = X(H_u(Y,Z)) - H_u(\nabla_X Y, Z) - H_u(Y, \nabla_X Z)$ for any $X, Y, Z \in \mathfrak{X}(M)$, where $H_u \equiv \nabla^2 u$. Locally we set $u_{ABC} = (\nabla^3 u)(T_A, T_B, T_C)$. Commutation formulae for u_{ABC} have been established by J.M. Lee, [30] (cf. also [12], p. 426) and are not needed through this paper. We shall use the divergence operator $\mathrm{div} : \mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$ determined by $\mathcal{L}_X \Psi_\theta = \mathrm{div}(X) \Psi_\theta$ for every $X \in \mathfrak{X}(M)$, where $\mathcal{L}_X$ is the Lie derivative. The horizontal gradient of $u \in C^1(M, \mathbb{R})$ is $\nabla^H u = \Pi_H \nabla u$ where $\Pi_H : T(M) \to H(M)$ is the projection associated to the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ and ∇u is the ordinary semi-Riemannian gradient of u with respect to g_θ i.e. $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. The sublaplacian of (M, θ) is the second order differential operator $\Delta_b u = -\mathrm{div}\left(\nabla^H u\right)$, $u \in C^2(M, \mathbb{R})$. Another useful expression of the sublaplacian is $\Delta_b u = -\mathrm{trace}_{G_\theta} \Pi_H \nabla^2 u$ or $\Delta_b u = -\sum_{a=1}^{2n} \{E_a(E_a(u)) - (\nabla_{E_a} E_a)(u)\}$ for any local G_θ-orthonormal frame $\{E_a : 1 \le a \le 2n\}$ of $H(M)$ on $U \subset M$. If $\{T_\alpha : 1 \le \alpha \le n\}$ is a local frame of $T_{1,0}(M)$ on $U \subset M$ then

$$\Delta_b u = -\nabla_\alpha u^\alpha - \nabla_{\overline{\alpha}} u^{\overline{\alpha}} \,. \tag{14}$$

A complex valued differential p-form $\omega \in \Omega^p(M) \otimes \mathbb{C}$ is a $(p,0)$-form (respectively a $(0,p)$-form) if $T_{0,1}(M) \rfloor \omega = 0$ (respectively $T_{0,1}(M) \rfloor \omega = 0$ and $T \rfloor \omega = 0$). Let $\Lambda^{p,0}(M) \to M$ and $\Lambda^{0,p}(M) \to M$ be the relevant bundles and $\Omega^{p,0}(M)$ and $\Omega^{0,p}(M)$ the corresponding spaces of sections. Let $\mathcal{F}$ be the flow on M tangent to the Reeb vector T (i.e. $T(\mathcal{F}) = \mathbb{R}T$). Let $\Omega^{1,0}_B(\mathcal{F}) = \{\omega \in \Omega^{1,0}(M) : T \rfloor \omega = 0\}$ be the space of all basic $(1,0)$-forms (on the foliated manifold $(M, \mathcal{F})$, cf. also [5]). If $\omega \in \Omega^{1,0}_B(\mathcal{F})$ one may use the Levi form to define a unique complex vector field $\omega^\sharp \in C^\infty(T_{0,1}(M))$. Here $\omega^\sharp$ is determined by $\omega(Z) = G_\theta(Z, \omega^\sharp)$ for any $Z \in T_{1,0}(M)$ hence locally $\omega^\sharp = \omega^{\overline{\beta}} T_{\overline{\beta}}$ where $\omega^{\overline{\beta}} = g^{\alpha\overline{\beta}} \omega_\alpha$ and $\omega = \omega_\alpha \theta^\alpha$. Let $\delta_b : \Omega^{1,0}_B(\mathcal{F}) \to C^\infty(M, \mathbb{C})$ be the differential operator (due to [30]) defined by $\delta_b \omega = \mathrm{div}\left(\omega^\sharp\right)$ and $\delta_b \theta = 0$ for any $\omega \in \Omega^{0,1}_B(\mathcal{F})$. Similarly if $\eta \in \Omega^{0,1}(M)$ then let $\eta^\sharp \in C^\infty(T_{1,0}(M))$ be determined by $\eta(\overline{Z}) = G_\theta(\eta^\sharp, \overline{Z})$, $Z \in T_{1,0}(M)$, and let us consider

$$\overline{\delta}_b : \Omega^{0,1}(M) \to C^\infty(M, \mathbb{C}), \quad \overline{\delta}_b \eta = \mathrm{div}\left(\eta^\sharp\right), \quad \eta \in \Omega^{0,1}(M),$$

so that (locally) $\eta^\sharp = \eta^\alpha T_\alpha$ where $\eta = \eta_{\overline{\beta}} \theta^{\overline{\beta}}$ and $\eta^\alpha = g^{\alpha\overline{\beta}} \eta_{\overline{\beta}}$. Also $\delta_b \omega = \nabla_{\overline{\beta}} \omega^{\overline{\beta}}$ and $\overline{\delta}_b \eta = \nabla_\alpha \eta^\alpha$. For each $f \in C^\infty(M, \mathbb{C})$ we set

$$(Pf)Z = g^{\alpha\overline{\beta}} \left(\nabla^3 f\right)(Z, T_\alpha, T_{\overline{\beta}}) + 2\, n\, i\, A\left(Z, (\nabla^H f)^{1,0}\right), \tag{15}$$

$$(Pf)\overline{Z} = 0, \quad (Pf)T = 0, \quad Z \in T_{1,0}(M).$$

Here $X^{1,0} = \Pi_{1,0}X$ for any $X \in H(M)$ and $\Pi_{1,0} : H(M) \otimes \mathbb{C} \to T_{1,0}(M)$ is the natural projection associated to $H(M) \otimes \mathbb{C} = T_{1,0}(M) \oplus T_{0,1}(M)$. Note that $g^{\alpha\overline{\beta}} \left(\nabla_{T_{\overline{\beta}}}(\nabla^2 f)\right)(T_\alpha, Z)$ is invariant under a transformation $T'_\alpha = U^\beta_\alpha T_\beta$ with $\det\left[U^\beta_\alpha\right] \neq 0$ on $U \cap U'$, hence $(Pf)Z$ is globally defined. Locally one has

$$Pf = \left(P_\beta f\right)\theta^\beta, \quad P_\beta f = f_\beta{}^{\overline{\alpha}}{}_{\overline{\alpha}} + 2ni\, A_{\beta\gamma} f^\gamma ,$$

(compare to Definition 1.1 and (1.2) in [10], p. 263). Similar to $P : C^\infty(M, \mathbb{C}) \to \Omega^{1,0}_B(\mathcal{F})$ we build $\overline{P} : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(M)$ given by

$$\left(\overline{P}f\right)\overline{Z} = g^{\alpha\overline{\beta}} \left(\nabla^3 f\right)(Z,\, T_{\overline{\beta}},\, T_\alpha) - 2\,n\,i\,A\left(\overline{Z},\, (\nabla^H f)^{0,1}\right), \tag{16}$$

$$(\overline{P}f)Z = 0, \quad (\overline{P}f)T = 0, \quad Z \in T_{1,0}(M),$$

where $X^{0,1} = \overline{X^{1,0}}$ for any $X \in H(M)$. Also let[1]

$$P_0 f = \delta_b(Pf) + \overline{\delta}_b(\overline{P}f), \quad f \in C^\infty(M, \mathbb{C}). \tag{17}$$

From now on we assume that M is a compact strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$. Then g_θ is a Riemannian metric on M. It should be observed that the operators above are complexifications of real operators familiar in Riemannian geometry, as follows. For instance let $\sharp$ be "raising of indices" with respect to g_θ i.e. $g_\theta\left(\alpha^\sharp, X\right) = \alpha(X)$ for any (real) 1-form $\eta \in \Omega^1(M)$ and any (real) vector field $X \in \mathfrak{X}(M)$. Then the musical isomorphisms $\sharp : \Omega^{1,0}_B(\mathcal{F}) \to C^\infty(T_{0,1}(M))$ and $\sharp : \Omega^{0,1}(M) \to C^\infty(T_{1,0}(M))$ (as built above) are restrictions of the $\mathbb{C}$-linear extension (to $\Omega^1(M) \otimes \mathbb{C} = C^\infty\left(T^*(M) \otimes \mathbb{C}\right)$) of $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ to $\Omega^{1,0}_B(\mathcal{F})$ and $\Omega^{0,1}(M)$ respectively. Also let $\Omega^1_B(\mathcal{F})$ be the space of all basic 1-forms on $(M, \mathcal{F})$ and $d_b : C^\infty(M) \to \Omega^1_B(\mathcal{F})$ the first order differential operator given by $d_b u = du - u_0\,\theta$ for every $u \in C^\infty(M, \mathbb{R})$ where $u_0 \equiv T(u)$. Let d^*_b be the formal adjoint of d_b i.e. $\left(d^*_b\omega\,,\, u\right)_{L^2} = (\omega\,,\, d_b u)_{L^2}$, $\omega \in \Omega^1_B(\mathcal{F})$, $u \in C^\infty(M)$, with respect to the L^2 inner products

$$(u, v)_{L^2} = \int_M uv\, \Psi_\theta\,, \quad (\alpha, \beta)_{L^2} = \int_M g^*_\theta(\alpha\,,\,\beta)\, \Psi_\theta\,,$$

for any $u, v \in C^\infty(M, \mathbb{R})$ and $\alpha, \beta \in \Omega^1(M)$. Let $d_b : C^\infty(M, \mathbb{C}) \to \Omega^1_B(\mathcal{F}) \otimes \mathbb{C}$ and $d^*_b : \Omega^1_B(\mathcal{F}) \otimes \mathbb{C} \to C^\infty(M, \mathbb{C})$ be the $\mathbb{C}$-linear extensions of d_b and d^*_b. Then

Lemma 1. i) $\Omega^1_B(\mathcal{F}) \otimes \mathbb{C} = \Omega^{1,0}_B(\mathcal{F}) \oplus \Omega^{0,1}(M)$, ii) $d_b f = \partial_b f + \overline{\partial}_b f$ *for any* $f \in C^\infty(M, \mathbb{C})$, iii) $d^*_b\big|_{\Omega^{1,0}_B(\mathcal{F})} = \partial^*_b = -\delta_b$, iv) $d^*_b\big|_{\Omega^{0,1}(M)} = \overline{\partial}^*_b = -\overline{\delta}_b$.

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[1] The operator P_0 in this paper and [10] differ by a multiplicative factor $\frac{1}{4}$.

Here the tangential C-R operator $\overline{\partial}_b$ is thought of as $\Omega^{0,1}(M)$-valued (i.e. one requests that $Z \rfloor \overline{\partial}_b f =$ and $T \rfloor \overline{\partial}_b f = 0$ to start with). Also $\partial_b f$ is the unique element of $\Omega_B^{1,0}(\mathcal{F})$ coinciding with df on $T_{1,0}(M)$. Locally $\partial_b f = f_\alpha \theta^\alpha$ and $\overline{\partial}_b f = f_{\overline{\alpha}} \theta^{\overline{\alpha}}$ where $f_\alpha \equiv T_\alpha(f)$ and $f_{\overline{\alpha}} \equiv T_{\overline{\alpha}}(f)$. Also ∂_b^* and $\overline{\partial}_b^*$ are the formal adjoints of $\partial_b : C^\infty(M, \mathbb{C}) \to \Omega_B^{1,0}(\mathcal{F})$ and $\overline{\partial}_b : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(M)$ with respect to the L^2 inner products

$$(f, g)_{L^2} = \int_M f \overline{g} \, \Psi_\theta , \quad (\omega_1 \, , \, \omega_2)_{L^2} = \int_M G_\theta^*(\omega_1 \, , \, \overline{\omega}_2) \, \Psi_\theta \, ,$$

for any $f, g \in C^\infty(M, \mathbb{C})$ and any complex 1-forms ω_1 , ω_2 either in $\Omega_B^{1,0}(\mathcal{F})$ or in $\Omega^{0,1}(M)$. Statements (i)-(ii) in Lemma 1 are immediate. The last equality in (iii) (respectively in (iv)) is due to [30] (cf. also [12], p. 280). To prove (iii) one integrates by parts in $(d_b^* \omega, f)_{L^2}$. For every $f \in C^\infty(M, \mathbb{R})$

$$\int_M g_\theta^* \Big((P + \overline{P}) f \, , \, \overline{d_b f} \Big) \; \Psi_\theta = \Big(P f + \overline{P} f \, , \, d_b f \Big)_{L^2} = - \left(P_0 f \, , \, f \right)_{L^2}$$

(compare to (1.3) in [10], p. 263). By a result of S-C. Chang & H-L. Chiu, [10], the operator P_0 is nonnegative i.e. $\int_M (P_0 u) u \, \Psi_\theta \geq 0$ for any $u \in C^\infty(M, \mathbb{R})$. We end the preparation of CR and pseudohermitian geometry by stating the identity (a straightforward consequence of (14))

$$u^\alpha \, u_\alpha{}^\beta{}_\beta + u^{\overline{\alpha}} \, u_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\beta}} = - u^\alpha \, P_\alpha u - u^{\overline{\alpha}} \, P_{\overline{\alpha}} u + \tag{18}$$

$$+ 2ni \left(A_{\alpha\beta} u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}} \right) - \left(\nabla^H u \right) (\Delta_b u).$$

Compare to (2.3) in [10], p. 267.

3. Bochner-Lichnerowicz formulae on Fefferman spaces

Let $S^1 \to C(M) \xrightarrow{\pi} M$ be the canonical circle bundle over a strictly pseudoconvex CR manifold M, of CR dimension n (cf. e.g. Definition 2.9 in [12], p. 119). We set $\mathfrak{M} = C(M)$ for simplicity. Let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on M and let F_θ be the corresponding Fefferman metric on $\mathfrak{M}$ i.e.

$$F_\theta = \pi^* \tilde{G}_\theta + 2 \, (\pi^* \theta) \odot \sigma, \tag{19}$$

$$\sigma = \frac{1}{n+2} \left\{ d\gamma + \pi^* \left(i \, \omega_\alpha{}^\alpha - \frac{i}{2} \, g^{\mu\overline{\nu}} \, dg_{\mu\overline{\nu}} - \frac{\rho}{4(n+1)} \, \theta \right) \right\} . \tag{20}$$

Cf. Definition 2.15 and Theorem 2.4 in [12], p. 128-129. As to the notations in (19)-(20) we set $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$ and $\tilde{G}_\theta(T, W) = 0$ for every $W \in \mathfrak{X}(M)$. Moreover γ is a local fibre coordinate on $\mathfrak{M}$. We recall that $F_\theta \in \mathrm{Lor}(\mathfrak{M})$ i.e. F_θ is a Lorentzian metric on $\mathfrak{M}$ (a semi-Riemannian metric of signature $(- + \cdots +)$).

Let D be the Levi-Civita connection of $(\mathfrak{M}, F_\theta)$. Given a point $z_0 \in \mathfrak{M}$ let $\{E_p : 1 \leq p \leq 2n+2\}$ be a local orthonormal (i.e. $F_\theta(E_p, E_q) = \epsilon_p \delta_{pq}$ with $\epsilon_p \in \{\pm 1\}$) frame of $T(\mathfrak{M})$, defined on an open neighborhood $\pi^{-1}(U) \subset \mathfrak{M}$ of z_0, such that $(D_{E_p} E_q)(z_0) = 0$ for any $1 \leq p, q \leq 2n+2$. Such a local frame may always be built by parallel translating a given orthonormal basis $\{e_p : 1 \leq p \leq 2n+2\} \subset T_{z_0}(\mathfrak{M})$ along the geodesics of $(\mathfrak{M}, F_\theta)$ issuing at z_0. Let $\Box$ be the wave operator (the Laplace-Beltrami operator of $(\mathfrak{M}, F_\theta)$). If $f \in C^\infty(\mathfrak{M}, \mathbb{R})$ and $g = F_\theta(Df, Df)$ then $\Box g = -\sum_{p=1}^{2n+2} \epsilon_p \left\{ E_p \left(E_p(g) \right) - \left(D_{E_p} E_p \right)(g) \right\}$. A calculation of $(\Box g)(z_0)$, merely adapting the proof of (G.IV.5) in [8], p. 131, to Lorentzian signature, leads to

$$-\frac{1}{2} \Box \left(F_\theta(Df, Df) \right) = F_\theta^* \left(D^2 f \, , \, D^2 f \right) - (Df)(\Box f) + \mathrm{Ric}_D(Df \, , \, Df). \tag{21}$$

Here $\mathrm{Ric}_D(X, Y) = \mathrm{trace}\left\{ Z \in T(\mathfrak{M}) \mapsto R^D(Z, Y)X \right\}$ and R^D is the curvature tensor field of D. Let us assume that M is a closed manifold (i.e. M is compact and $\partial M = \emptyset$). Then $\mathfrak{M}$ is a closed manifold, as well (as the total space of a locally trivial bundle over a compact manifold, with compact fibres). Integration of (21) over $\mathfrak{M}$ leads (by Green's lemma) to the (Lorentzian analog to the) L^2 Bochner-Lichnerowicz formula

$$\int_{\mathfrak{M}} \left\{ F_\theta^* \left(D^2 f \, , \, D^2 f \right) + \mathrm{Ric}_D(Df \, , \, Df) - (Df)(\Box f) \right\} d\,\mathrm{vol}(F_\theta) = 0. \tag{22}$$

4. Curvature theory

By a result in [21] the 1-form $\sigma \in \Omega^1(M)$ is a connection form in the canonical circle bundle $S^1 \to \mathfrak{M} \to M$. Let $X^\uparrow \in \mathfrak{X}(\mathfrak{M})$ denote the horizontal lift of $X \in \mathfrak{X}(M)$ i.e. $X^\uparrow_z \in \mathrm{Ker}(d_z \pi)$ and $(d_z \pi) X^\uparrow_z = X_{\pi(z)}$ for any $z \in \mathfrak{M}$. Let $S \in \mathfrak{X}(\mathfrak{M})$ be the tangent to the S^1-action i.e. locally $S = [(n+2)/2] \, \partial / \partial \gamma$. The Levi-Civita connection D of $(\mathfrak{M}, F_\theta)$ is given by (cf. Lemma 2 in [6], p. 03504-26)

$$D_{X^\uparrow} Y^\uparrow = (\nabla_X Y)^\uparrow + \{\Omega(X, Y) \circ \pi\} \; T^\uparrow + \left\{ \sigma \left(\left[X^\uparrow, Y^\uparrow \right] \right) - 2 \, A(X, Y) \circ \pi \right\} S, \tag{23}$$

$$D_{X^\uparrow} T^\uparrow = \{\tau(X) + \phi(X)\}^\uparrow, \tag{24}$$

$$D_{T^\uparrow} X^\uparrow = (\nabla_T X + \phi X)^\uparrow + 4 \, (d\sigma)(X^\uparrow, T^\uparrow) \, S, \tag{25}$$

$$D_{X^\uparrow} S = D_S X^\uparrow = \frac{1}{2} (JX)^\uparrow, \tag{26}$$

$$D_{T^\uparrow} T^\uparrow = 2 \, V^\uparrow, \quad D_S S = D_S T^\uparrow = D_{T^\uparrow} S = 0, \tag{27}$$

where $\Omega = -d\theta$ while $\phi : H(M) \to H(M)$ and $V \in H(M)$ are the bundle endomorphism and vector field determined by

$$G_\theta(\phi X, Y) \circ \pi = (d\sigma)(X^\uparrow\,,\,Y^\uparrow), \quad G_\theta(V, X) = (d\sigma)(T^\uparrow\,,\,X^\uparrow), \tag{28}$$

for any $X, Y \in H(M)$. Locally ϕ and V are given by

$$\phi_\alpha{}^\beta = \frac{i}{2(n+2)}\left\{R_\alpha{}^\beta - \frac{\rho}{2(n+1)}\,\delta_\alpha^\beta\right\}, \quad \phi_\alpha{}^{\overline{\beta}} = 0, \quad \phi_\alpha{}^0 = 0, \tag{29}$$

$$V^\alpha = g^{\alpha\overline{\beta}} V_{\overline{\beta}}\,, \quad V_{\overline{\beta}} = \frac{1}{2(n+2)}\left\{\frac{1}{4(n+1)}\,\rho_{\overline{\beta}} + i\,W^\alpha_{\alpha\overline{\beta}}\right\}. \tag{30}$$

In particular $[J, \phi] = 0$ (as a consequence of (29)). We recall (cf. (1.100) in [12], p. 58)

$$\mathrm{Ric}_{g_\theta}(T_\mu\,,\,T_{\overline{\nu}}) = -\frac{1}{2}\,R_{\mu\overline{\nu}} + g_{\mu\overline{\nu}}\,, \tag{31}$$

$$R_{\mu\nu} = i(n-1)\,A_{\mu\nu}\,, \quad R_{0\nu} = S^{\overline{\mu}}_{\overline{\mu}\nu}\,, \quad R_{\mu 0} = 0, \quad R_{00} = 0. \tag{32}$$

Here Ric_{g_θ} is the Ricci curvature of (M, g_θ). Also $S(X, Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X$ for any $X, Y \in \mathfrak{X}(M)$, so that $S^{\overline{\mu}}_{\overline{\mu}\nu}$ are among $S^{\,j}_{k\ell} T_j = S(T_k\,,\,T_\ell)$. As a consequence of (31) one has $R_{\mu\overline{\nu}} = R_{\overline{\nu}\mu}$. Take the derivative of (20)

$$(n+2)\,d\sigma = \pi^*\left\{i d\omega_\alpha{}^\alpha - \frac{i}{2}\,dg^{\mu\overline{\nu}} \wedge dg_{\mu\overline{\nu}} - \frac{1}{4(n+1)}\,d(\rho\theta)\right\}$$

and observe that $dg^{\mu\overline{\nu}} \wedge dg_{\mu\overline{\nu}} = 0$. Also (by Theorem 1.7 in [12], p. 55)

$$d\omega_\alpha{}^\alpha = R_{\mu\overline{\nu}}\,\theta^\mu \wedge \theta^{\overline{\nu}} + \left(W^\alpha_{\alpha\lambda}\,\theta^\lambda - W^\alpha_{\alpha\overline{\mu}}\,\theta^{\overline{\mu}}\right) \wedge \theta\,.$$

By (31)-(32)

$$\mathrm{Ric}_\nabla(X, JY) = -2i\,\left(R_{\mu\overline{\nu}}\theta^\mu \wedge \theta^{\overline{\nu}}\right)(X, Y) - (n-1)A(X, Y) \tag{33}$$

for any $X, Y \in H(M)$. Also $d(\rho\theta) = -\rho\,\Omega$ on $H(M) \otimes H(M)$. Consequently

$$2(d\sigma)(X^\uparrow, Y^\uparrow) = \frac{1}{n+2}\left\{\frac{\rho}{2(n+1)}\,\Omega(X, Y) - \right. \tag{34}$$
$$\left. - (n-1)\,A(X, Y) - \mathrm{Ric}_\nabla(X, JY)\right\}.$$

By a result in [28], Vol. I, p. 65, $[X, Y]^\uparrow$ is the horizontal component of $\left[X^\uparrow, Y^\uparrow\right]$ for any $X, Y \in \mathfrak{X}(M)$. When $X, Y \in H(M)$ the vertical component may be easily derived from (34). One obtains the decomposition

$$\left[X^\uparrow\,,\,Y^\uparrow\right] = [X, Y]^\uparrow + \frac{2}{n+2}\left\{\mathrm{Ric}_\nabla(X, JY) + \right. \tag{35}$$
$$\left. + (n-1)\,A(X, Y) - \frac{\rho}{2(n+1)}\,\Omega(X, Y)\right\}\,S\,.$$

Similarly let us compute $f \in C^\infty(M)$ in $[X^\uparrow, T^\uparrow] = [X, T]^\uparrow + fS$. If $\varphi = i\left(W^\alpha_{\alpha\lambda}\theta^\lambda - W^\alpha_{\alpha\overline{\mu}}\theta^{\overline{\mu}}\right)$ then

$$i\,(d\omega_\alpha{}^\alpha)(X, T) = (\varphi \wedge \theta)(X, T) = \frac{1}{2}\,\varphi(X),$$

$$2(n+2)(d\sigma)(X^\uparrow, T^\uparrow) = \varphi(X) - \frac{1}{2(n+1)}\,d(\rho\theta)(X, T)$$

or

$$2(d\sigma)(X^\uparrow, T^\uparrow) = \frac{1}{n+2}\left\{\varphi(X) - \frac{1}{4(n+1)}\,X(\rho)\right\} \tag{36}$$

as $T \rfloor d\theta = 0$. We conclude (as $\sigma(S) = \frac{1}{2}$)

$$[X^\uparrow, T^\uparrow] = [X, T]^\uparrow + \frac{2}{n+2}\left\{\frac{1}{4(n+1)}\,X(\rho) - \varphi(X)\right\}S. \tag{37}$$

Lemma 2. *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and $\theta \in \mathcal{P}_+$ a positively oriented contact form. The curvature R^D of the Lorentzian manifold $(\mathfrak{M}, F_\theta)$ is given by*

$$\begin{aligned}
R^D(X^\uparrow, Y^\uparrow)Z^\uparrow = {} & \left(R^\nabla(X, Y)Z\right)^\uparrow - \\
& - \frac{1}{2(n+1)(n+2)}\,\{X(\rho)\,\Omega(Y, Z) - Y(\rho)\,\Omega(X, Z)\}\,S - \\
& - \frac{n+5}{n+2}\{(\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z)\}\,S + \\
& + \frac{1}{n+2}\{(\nabla_X \mathrm{Ric}_\nabla)(Y, JZ) - (\nabla_Y \mathrm{Ric}_\nabla)(Y, JZ)\}\,S + \\
& + \Omega(Y, Z)\left\{(\tau X)^\uparrow + (\phi X)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JX)^\uparrow\right\} - \\
& - \Omega(X, Z)\left\{(\tau Y)^\uparrow + (\phi Y)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JY)^\uparrow\right\} + \\
& + \frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(Y, JZ) - (n+5)\,A(Y, Z)\}\,(JX)^\uparrow - \\
& - \frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(X, JZ) - (n+5)\,A(X, Z)\}\,(JY)^\uparrow - \\
& - \frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X, JY)\,(JZ)^\uparrow - 2\Omega(X, Y)\,\mathrm{Ric}_\nabla(T, JZ)\,S\right\} - \\
& - \frac{1}{n+2}\left\{(n-1)A(X, Y) - \frac{\rho}{2(n+1)}\,\Omega(X, Y)\right\}(JZ)^\uparrow - \\
& - 2\,\Omega(X, Y)\left\{(\phi Z)^\uparrow + \frac{2}{n+2}\left[\varphi(Z) - \frac{1}{4(n+1)}\,Z(\rho)\right]S\right\}.
\end{aligned} \tag{38}$$

$$R^D(X^\uparrow, T^\uparrow)Z^\uparrow = \left(R^\nabla(X,T)Z\right)^\uparrow + ((\nabla_X\phi)Z)^\uparrow + \tag{39}$$

$$+\frac{1}{n+2}\left\{\varphi(Z)(JX)^\uparrow + \varphi(X)(JZ)^\uparrow - \left[\mathrm{Ric}_\nabla(X,J\phi Z) + \mathrm{Ric}_\nabla(\tau X, JZ)\right]S\right\} -$$

$$-\frac{1}{4(n+1)(n+2)}\left\{Z(\rho)(JX)^\uparrow + X(\rho)(JZ)^\uparrow\right\} +$$

$$+\frac{2}{n+2}\left\{(\nabla_X\varphi)Z - \frac{1}{4(n+1)}(\nabla_X d\rho)Z\right\}S -$$

$$-\frac{1}{n+2}\left\{(\nabla_T \mathrm{Ric}_\nabla)(X,JZ) - (n+5)(\nabla_T A)(X,Z)\right\}S +$$

$$+\left\{\Omega(X,\phi Z) - \Omega(\tau X, Z)\right\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}S\right\} -$$

$$-2\Omega(X,Z)\left\{V^\uparrow - \frac{T(\rho)}{4(n+1)(n+2)}S\right\} - \frac{3(n+3)}{n+2}\left\{A(X,\phi Z) - A(\tau X, Z)\right\}S,$$

$$R^D(X^\uparrow, S)Z^\uparrow = -\frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla(X,Z) + (n+5)\,A(X,JZ)\right\}S - \tag{40}$$

$$-\frac{1}{2}G_\theta(X,Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}S\right\},$$

$$R^D(X^\uparrow, Y^\uparrow)T^\uparrow = ((\nabla_X\tau)Y + (\nabla_X\phi)Y)^\uparrow + 4\Omega(X,Y)V^\uparrow - \tag{41}$$

$$-\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(J\tau X, Y) - \mathrm{Ric}_\nabla(X, J\tau Y) + \mathrm{Ric}_\nabla(J\phi X, Y) - \mathrm{Ric}_\nabla(X, J\phi Y)\right\}S -$$

$$-\frac{n+5}{2(n+2)^2}\left\{\mathrm{Ric}_\nabla(\tau X, JY) - \mathrm{Ric}_\nabla(JX, \tau Y) + 2(n-1)\Omega(\tau X, \tau Y)\right\},$$

$$R^D(X^\uparrow, Y^\uparrow)S = 0, \quad R^D(T^\uparrow, S)T^\uparrow = 0, \quad R^D(T^\uparrow, S)S = 0, \tag{42}$$

$$R^D(T^\uparrow, S)Z^\uparrow = \tag{43}$$

$$= \frac{1}{n+2}\left\{\varphi(JZ) - 2\,\varphi(Z) - \frac{1}{4(n+1)}\left[(JZ)(\rho) - 2\,Z(\rho)\right]\right\}S,$$

for any $X, Y, Z \in H(M)$.

Proof. As $H(M)$ is parallel with respect to ∇ one has $\nabla_Y Z \in H(M)$. Then (by (23) and (34))

$$D_{X^\uparrow}(\nabla_Y Z)^\uparrow = (\nabla_X \nabla_Y Z)^\uparrow + \Omega(X, \nabla_Y Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}S\right\} + \tag{44}$$

$$+\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X, J\nabla_Y Z) - (n+5)\,A(X, \nabla_Y Z)\right\}S.$$

Next (by (23)-(24), (26), (34) and (44))

$$D_{X^\uparrow} D_{Y^\uparrow} Z^\uparrow = (\nabla_X \nabla_Y Z)^\uparrow + \tag{45}$$

$$+\{X(\Omega(Y,Z))+\Omega(X,\nabla_Y Z)\}\left\{T^\uparrow-\frac{\rho}{2(n+1)(n+2)}\,S\right\}+$$

$$-\frac{X(\rho)}{2(n+1)(n+2)}\,\Omega(Y,Z)S\;-\frac{n+5}{n+2}\,\{X(A(Y,Z))+A(X,\nabla_Y Z)\}\,S+$$

$$+\frac{1}{n+2}\,\{X(\mathrm{Ric}_\nabla(Y,JZ))+\mathrm{Ric}_\nabla(X,J\nabla_Y Z)\}\,S+$$

$$+\Omega(Y,Z)\left\{(\tau X)^\uparrow+(\phi X)^\uparrow-\frac{\rho}{4(n+1)(n+2)}\,(JX)^\uparrow\right\}+$$

$$+\frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(Y,JZ)-(n+5)\,A(Y,Z)\}\,(JX)^\uparrow\,.$$

The calculation of $D_{[X^\uparrow,Y^\uparrow]}Z^\uparrow$ is a bit trickier as $[X,Y]\notin H(M)$ in general. To start with one uses the decomposition (35) followed by $[X,Y]=\Pi_H[X,Y]+\theta([X,Y])T$. This yields (by (26))

$$D_{[X^\uparrow,Y^\uparrow]}Z^\uparrow=D_{[X,Y]^\uparrow}Z^\uparrow+\frac{2}{n+2}\,B(X,Y)\,D_S Z^\uparrow=$$

$$=D_{(\Pi_H[X,Y])^\uparrow}Z^\uparrow+\theta([X,Y])\,D_{T^\uparrow}Z^\uparrow+\frac{1}{n+2}\,B(X,Y)\,(JZ)^\uparrow$$

where we have set

$$B(X,Y)=\mathrm{Ric}_\nabla(X,JY)+(n-1)\,A(X,Y)-\frac{\rho}{2(n+1)}\,\Omega(X,Y)$$

for simplicity. At this point we may use (23) (as $\Pi_H[X,Y]\in H(M)$) and (25) so that

$$D_{[X^\uparrow,Y^\uparrow]}Z^\uparrow=(\nabla_{\Pi_H[X,Y]}Z)^\uparrow+\Omega(\Pi_H[X,Y],Z)\,T^\uparrow-$$

$$-2\left\{(d\sigma)\left((\Pi_H[X,Y])^\uparrow\,,\,Z^\uparrow\right)+A\,(\Pi_H[X,Y]\,,\,Z)\right\}S+$$

$$+\theta([X,Y])\left\{(\nabla_T Z)^\uparrow+(\phi Z)^\uparrow+4(d\sigma)(Z^\uparrow\,,\,T^\uparrow)S\right\}+\frac{1}{n+2}\,B(X,Y)\,(JZ)^\uparrow\,.$$

Next (by $T\rfloor\Omega=T\rfloor A=0$ and the identities (34) and (36))

(46) $$D_{[X^\uparrow,Y^\uparrow]}Z^\uparrow=(\nabla_{[X,Y]}Z)^\uparrow+$$

$$+\Omega([X,Y],Z)\left\{T^\uparrow-\frac{\rho}{2(n+1)(n+2)}\,S\right\}-\frac{n+5}{n+2}\,A([X,Y],Z)S+$$

$$+\frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X,JY)\,(JZ)^\uparrow+\mathrm{Ric}_\nabla\,(\Pi_H[X,Y]\,,\,JZ)\,S\right\}+$$

$$+\frac{1}{n+2}\left\{(n-1)A(X,Y)-\frac{\rho}{2(n+1)}\,\Omega(X,Y)\right\}\,(JZ)^\uparrow+$$

$$+\theta([X,Y])\left\{(\phi Z)^\uparrow+\frac{2}{n+2}\left[\varphi(Z)-\frac{1}{4(n+1)}\,Z(\rho)\right]S\right\}.$$

Moreover (by (45)-(46))

$$
\begin{aligned}
(47)\qquad R^D(X^\uparrow\,,\,Y^\uparrow)Z^\uparrow &= \left([D_{X^\uparrow}\,,\,D_{Y^\uparrow}] - D_{[X^\uparrow\,,\,Y^\uparrow]}\right)Z^\uparrow = (\nabla_X\nabla_Y Z)^\uparrow + \\
&+ \{X(\Omega(Y,Z)) + \Omega(X,\nabla_Y Z)\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\,S\right\} - \\
&- \frac{X(\rho)}{2(n+1)(n+2)}\,\Omega(Y,Z)S - \frac{n+5}{n+2}\,\{X(A(Y,Z)) + A(X,\nabla_Y Z)\}\,S + \\
&+ \frac{1}{n+2}\,\{X(\mathrm{Ric}_\nabla(Y,JZ)) + \mathrm{Ric}_\nabla(X,J\nabla_Y Z)\}\,S + \\
&+ \Omega(Y,Z)\left\{(\tau X)^\uparrow + (\phi X)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JX)^\uparrow\right\} + \\
&+ \frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(Y,JZ) - (n+5)\,A(Y,Z)\}\,(JX)^\uparrow - (\nabla_Y\nabla_X Z)^\uparrow - \\
&- \{Y(\Omega(X,Z)) + \Omega(Y,\nabla_X Z)\}\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\,S\right\} + \\
&+ \frac{Y(\rho)}{2(n+1)(n+2)}\,\Omega(X,Z)S + \frac{n+5}{n+2}\,\{Y(A(X,Z)) + A(Y,\nabla_X Z)\}\,S - \\
&- \frac{1}{n+2}\,\{Y(\mathrm{Ric}_\nabla(X,JZ)) + \mathrm{Ric}_\nabla(Y,J\nabla_X Z)\}\,S - \\
&- \Omega(X,Z)\left\{(\tau Y)^\uparrow + (\phi Y)^\uparrow - \frac{\rho}{4(n+1)(n+2)}\,(JY)^\uparrow\right\} - \\
&- \frac{1}{2(n+2)}\,\{\mathrm{Ric}_\nabla(X,JZ) - (n+5)\,A(X,Z)\}\,(JY)^\uparrow - (\nabla_{[X,Y]}Z)^\uparrow - \\
&- \Omega([X,Y],Z)\left\{T^\uparrow - \frac{\rho}{2(n+1)(n+2)}\,S\right\} + \frac{n+5}{n+2}\,A([X,Y],Z)S - \\
&- \frac{1}{n+2}\left\{\mathrm{Ric}_\nabla(X,JY)\,(JZ)^\uparrow + \mathrm{Ric}_\nabla\,(\Pi_H[X,Y]\,,\,JZ)\,S\right\} - \\
&- \frac{1}{n+2}\left\{(n-1)A(X,Y) - \frac{\rho}{2(n+1)}\,\Omega(X,Y)\right\}\,(JZ)^\uparrow - \\
&- \theta([X,Y])\left\{(\phi Z)^\uparrow + \frac{2}{n+2}\left[\varphi(Z) - \frac{1}{4(n+1)}\,Z(\rho)\right]S\right\}.
\end{aligned}
$$

Using the identity

$$
(48)\qquad [X,Y] = \nabla_X Y - \nabla_Y X + 2\Omega(X,Y)T, \quad X,Y \in H(M),
$$

one has

$$
X(\Omega(Y,Z)) + \Omega(X,\nabla_Y X) - Y(\Omega(X,Z)) - \Omega(Y,\nabla_X Z) - \Omega([X,Y],Z) = 0
$$

as $\nabla\Omega = 0$ and $T \rfloor \Omega = 0$. Similarly (again by (47) and $T \rfloor A = 0$)

$$
-X(A(Y,Z)) - A(X,\nabla_Y Z) + Y(A(X,Z)) + A(Y,\nabla_X Z) + A([X,Y],Z) =
$$

$$= -(\nabla_X A)(Y,Z) + (\nabla_Y A)(X,Z).$$

Next (by $\nabla J = 0$)

$$X(\mathrm{Ric}_\nabla(Y, JZ)) + \mathrm{Ric}_\nabla(X, J\nabla_Y Z) -$$
$$-Y(\mathrm{Ric}_\nabla(X, JZ)) - \mathrm{Ric}_\nabla(Y, J\nabla_X Z) - \mathrm{Ric}_\nabla(\Pi_H[X,Y], JZ) =$$
$$= (\nabla_X \mathrm{Ric}_\nabla)(Y, JZ) - (\nabla_Y \mathrm{Ric}_\nabla)(Y, JZ) + 2\Omega(X,Y)\,\mathrm{Ric}_\nabla(T, JZ).$$

Consequently (47) yields (38). The remaining identities (39)-(43) may be proved in a similar manner.

Using Lemma 2 one may compute the Ricci curvature of $(\mathfrak{M}, F_\theta)$. Let $\{E_a : 1 \le a \le 2n\}$ be an orthonormal frame of $H(M)$ i.e. $G_\theta(E_a, E_b) = \delta_{ab}$. Then $\left\{\tilde{E}_p : 1 \le p \le 2n+2\right\} \equiv \{E_a^\uparrow,\ T^\uparrow \pm S\ :\ 1 \le a \le 2n\}$ with $\tilde{E}_a = E_a^\uparrow$, $\tilde{E}_{2n+1} = T^\uparrow - S$ and $\tilde{E}_{2n+2} = T^\uparrow + S$, is a local F_θ-orthonormal frame of $T(\mathfrak{M})$, so that $\mathrm{Ric}_D(U,W) = \sum_{p=1}^{2n+2} \epsilon_p F_\theta\left(R^D(\tilde{E}_p\,,\ W)U\,,\ \tilde{E}_p\right)$ i.e.

$$\mathrm{Ric}_D(U,W) = \sum_{a=1}^{2n} F_\theta\left(R^D(E_a^\uparrow\,,\ W)U\,,\ E_a^\uparrow\right) + \tag{49}$$
$$+2\left\{F_\theta\left(R^D(T^\uparrow, W)U\,,\ S\right) + F_\theta\left(R^D(S, W)U\,,\ T^\uparrow\right)\right\}$$

for any $U, W \in \mathfrak{X}(\mathfrak{M})$. We may state the following

Lemma 3. *For any $X, Y \in H(M)$*

$$\mathrm{Ric}_D(X^\uparrow, Y^\uparrow) = \frac{n+1}{n+2}\{\mathrm{Ric}_\nabla(X,Y) + 3\,A(X, JY)\} + \tag{50}$$
$$+\frac{\rho}{2(n+1)(n+2)}\,G_\theta(X,Y),$$

$$\mathrm{Ric}_D(X^\uparrow, T^\uparrow) = \mathrm{Ric}_\nabla(X,T) + \mathrm{trace}\ \{\Pi_H(\nabla\phi)X\} + \tag{51}$$
$$+\frac{1}{n+2}\,\varphi(JX) - 2\,\Omega(V,X) + \frac{1}{4(n+1)(n+2)}\,\Omega(X\,,\ \nabla^H\rho),$$

$$\mathrm{Ric}_D(X^\uparrow, S) = 0, \tag{52}$$

$$\mathrm{Ric}_D(T^\uparrow, T^\uparrow) = \frac{1}{n+2}\,\mathrm{trace}\left\{\frac{\rho}{4(n+1)}\,J\phi - 3(n+3)\,\tau^2\right\} + \tag{53}$$
$$+\frac{1}{n+2}\,\mathrm{trace}_{G_\theta}\Pi_H\,\{\mathrm{Ric}_\nabla(\cdot\,,\ J\phi\,\cdot) + \mathrm{Ric}_\nabla(\tau\,\cdot\,,\ J\,\cdot) -$$
$$\left.-\nabla\varphi + \frac{1}{4(n+1)}\,\nabla d\rho + \frac{n+5}{2}\,\nabla_T A - \frac{1}{2}\,(\nabla_T \mathrm{Ric}_\nabla)(\cdot\,,\ J\,\cdot)\right\},$$

$$\mathrm{Ric}_D(T^\uparrow, S) = \frac{\rho}{4(n+1)}\,, \quad \mathrm{Ric}_D(S,S) = \frac{n}{2}\,. \tag{54}$$

Proof. Let $X, Y, E \in H(M)$ and let us replace (X, Y, Z) in (38) by (E, Y, X) and take the inner product of the resulting identity with $E^\uparrow$. As

$$F_\theta(X^\uparrow\,,\,Y^\uparrow) = G_\theta(X, Y)\circ\pi, \quad F_\theta(X^\uparrow\,,\,T^\uparrow) = 0, \quad F_\theta(X^\uparrow\,,\,S) = 0,$$

and $G_\theta(JX, JY) = G_\theta(X, Y)$ we obtain

$$F_\theta\left(R^D(E^\uparrow, Y^\uparrow)X^\uparrow, E^\uparrow\right) = G_\theta\left(R^\nabla(E, Y)X, E\right) +$$
$$+\Omega(Y, X)\left\{G_\theta(\tau E, E) + G_\theta(\phi E, E)\right\} -$$
$$-\Omega(E, X)\left\{G_\theta(\tau Y, E) + G_\theta(\phi Y, E) - \frac{\rho}{4(n+1)(n+2)}\,G_\theta(JY, E)\right\} -$$
$$-\frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla(E, JX) - (n+5)\,A(E, X)\right\}G_\theta(JY, E) -$$
$$-\frac{1}{n+2}\,\mathrm{Ric}_\nabla(E, JY)\,G_\theta(JX, E) -$$
$$-\frac{1}{n+2}\left\{(n-1)A(E, Y) - \frac{\rho}{2(n+1)}\,\Omega(E, Y)\right\}G_\theta(JX, E).$$

Let us replace E by E_a and sum over $1 \le a \le 2n$. Since $\mathrm{trace}(\tau) = 0$ one obtains

$$\sum_a F_\theta\left(R^D(E_a^\uparrow\,,\,Y^\uparrow)X^\uparrow\,,\,E_a^\uparrow\right) = \mathrm{Ric}_\nabla(X, Y) + \tag{55}$$
$$+\Omega(Y, X)\,\mathrm{trace}(\phi) - \Omega(\tau Y, X) - \Omega(\phi Y, X) + \frac{\rho}{4(n+1)(n+2)}\,\Omega(JY, X) -$$
$$-\frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla(JY, JX) - (n+5)\,A(JY, X)\right\} - \frac{1}{n+2}\,\mathrm{Ric}_\nabla(JX, JY) -$$
$$-\frac{1}{n+2}\left\{(n-1)A(JX, Y) - \frac{\rho}{2(n+1)}\,\Omega(JX, Y)\right\}.$$

Note that (by the symmetry of A together with $\tau\circ J + J\circ\tau = 0$)

$$A(JX, Y) = A(X, JY), \quad \Omega(\tau Y, X) = A(X, JY).$$

To further simplify (55) we need some preparation. Let us replace X by JX in (33). One has

$$\mathrm{Ric}_\nabla(JX, JY) = -2i\left(R_{\mu\bar\nu}\theta^\mu\wedge\theta^{\bar\nu}\right)(JX, Y) - (n-1)\,A(JX, Y) =$$
$$= 2i\left(R_{\mu\bar\nu}\theta^\mu\wedge\theta^{\bar\nu}\right)(Y, JX) - (n-1)\,A(X, JY) =$$

(by applying (33) once again)

$$= -\mathrm{Ric}_\nabla(Y, J^2X) - (n-1)\,A(Y, JX) - (n-1)\,A(X, JY)$$

or (as $J^2 = -I$ on $H(M)$)

$$\mathrm{Ric}_\nabla(JX, JY) = \mathrm{Ric}_\nabla(X, Y) - 2(n-1)\,A(X, JY) \tag{56}$$

for any $X, Y \in H(M)$. Here we have also used the symmetry of Ric_∇ on $H(M) \otimes H(M)$ i.e. $\mathrm{Ric}_\nabla(X, Y) = \mathrm{Ric}_\nabla(Y, X)$ which is an immediate consequence of (31)-(32). Moreover $\mathrm{trace}(\phi) = 0$ as a corollary of (29) and the fact that the trace of the endomorphism $\phi : H(M) \to H(M)$ coincides with the trace of its extension by $\mathbb{C}$-linearity to $H(M)\otimes\mathbb{C}$ (and $\phi_\alpha{}^\beta$ is purely imaginary). Next one needs to compute $\Omega(\phi Y, X)$. If $\{T_\alpha : 1 \leq \alpha \leq n\}$ is a local frame of $T_{1,0}(M)$ and $X = X^\alpha T_\alpha + X^{\overline{\alpha}} T_{\overline{\alpha}}$ for some $X^\alpha \in C^\infty(U, \mathbb{C})$ (with $\overline{X^\alpha} = X^{\overline{\alpha}}$) then (by (29))

$$\Omega(\phi Y\,,\,X) = \tag{57}$$

$$= \frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla\left(Y^{1,0}\,,\,X^{0,1}\right) + \mathrm{Ric}_\nabla\left(Y^{0,1}\,,\,X^{1,0}\right)\right\} -$$

$$-\frac{\rho}{4(n+1)(n+2)}\left\{G_\theta\left(Y^{1,0}\,,\,X^{0,1}\right) + G_\theta\left(Y^{0,1}\,,\,X^{1,0}\right)\right\}$$

where we have set $X^{1,0} = X^\alpha T_\alpha$ and $X^{0,1} = \overline{X^{1,0}}$ (so that $X = X^{1,0} + X^{0,1}$). To further compute (57) let us observe that (by (32))

$$\mathrm{Ric}_\nabla\left(Y^{1,0}\,,\,X^{0,1}\right) + \mathrm{Ric}_\nabla\left(Y^{0,1}\,,\,X^{1,0}\right) =$$

$$= \mathrm{Ric}_\nabla(X, Y) - i(n-1)\left\{A\left(Y^{1,0}\,,\,X^{1,0}\right) - A\left(Y^{0,1}\,,\,X^{0,1}\right)\right\} =$$

(as A vanishes on $T_{1,0}(M)\otimes T_{0,1}(M)$, a consequence of $\tau\, T_{1,0}(M) \subset T_{0,1}(M)$)

$$= \mathrm{Ric}_\nabla(X, Y) - i(n-1)\left\{A\left(Y^{1,0}\,,\,X\right) - A\left(Y^{0,1}\,,\,X\right)\right\}$$

or (as $JY = i(Y^{1,0} - Y^{0,1})$)

$$\mathrm{Ric}_\nabla\left(Y^{1,0}\,,\,X^{0,1}\right) + \mathrm{Ric}_\nabla\left(Y^{0,1}\,,\,X^{1,0}\right) = \mathrm{Ric}_\nabla(X, Y) - (n-1)\,A(X, JY). \tag{58}$$

Substitution from (58) into (57) leads to

$$\Omega(\phi Y, X) = \frac{1}{2(n+2)}\{\mathrm{Ric}_\nabla(X, Y) - (n-1)A(X, JY)\} - \tag{59}$$

$$-\frac{\rho}{4(n+1)(n+2)}\,G_\theta(X, Y)$$

for any $X, Y \in H(M)$. Substitution from (56) and (59) into (55) leads to

$$\sum_{a=1}^{2n} F_\theta\left(R^D(E_a^\uparrow\,,\,Y^\uparrow)X^\uparrow\,,\,E_a^\uparrow\right) = \tag{60}$$

$$= \frac{n}{n+2}\,\mathrm{Ric}_\nabla(X, Y) + \frac{2(n-1)}{n+2}\,A(X, JY) + \frac{\rho}{(n+1)(n+2)}\,G_\theta(X, Y).$$

Let us take the inner product of (39) with S and use

$$F_\theta(S, S) = 0, \quad F_\theta(T^\uparrow, S) = \frac{1}{2}, \quad F_\theta(X^\uparrow, S) = 0, \quad X \in H(M).$$

Since (by (39))

$$R^D\left(X^\uparrow, T^\uparrow\right)Z^\uparrow \equiv \{\Omega(X,\phi Z) - \Omega(\tau X, Z)\}T^\uparrow, \quad \bmod H(M)^\perp,\ S,$$

we obtain

$$F_\theta(R^D(X^\uparrow, T^\uparrow)Z^\uparrow\,,\,S) = \frac{1}{2}\{\Omega(X,\phi Z) - \Omega(\tau X, Z)\}. \tag{61}$$

Therefore the last two terms in (49) (with $U = X^\uparrow$ and $W = Y^\uparrow$) may be computed (by (61) and (59)) as

$$F_\theta\left(R^D(T^\uparrow, Y^\uparrow)X^\uparrow, S\right) + F_\theta\left(R^D(S, Y^\uparrow)X^\uparrow, T^\uparrow\right) = \tag{62}$$

$$= \frac{1}{2(n+2)}\{\mathrm{Ric}_\nabla(X,Y) + (n+5)\,A(X,JY)\} - \frac{\rho}{4(n+1)(n+2)}\,G_\theta(X,Y).$$

Finally formulae (49) and (62) lead to (50). The remaining identities (51)-(54) may be proved in a similar manner.

5. Pseudohermitian Bochner-Lichnerowicz formula

Let $f \in C^\infty(\mathfrak{M})$. Then $Df = \sum_{j=1}^{2n+2} \epsilon_j \tilde{E}_j(f)\tilde{E}_j$ hence

$$D(u\circ\pi) = \sum_a E_a(u)E_a^\uparrow + 2T(u)S = \left(\nabla^H u\right)^\uparrow + 2u_0\,S \tag{63}$$

for any $u \in C^\infty(M)$, where $u_0 = T(u)$. Next (by (50), (52) and (54))

$$\mathrm{Ric}_D(D(u\circ\pi)\,,\,D(u\circ\pi)) = 2nu_0^2 + \tag{64}$$

$$+\frac{n+1}{n+2}\left\{\mathrm{Ric}_\nabla\left(\nabla^H u, \nabla^H u\right) + 3A(\nabla^H u\,,\,J\nabla^H u)\right\} + \frac{\rho}{2(n+1)(n+2)}\,\left\|\nabla^H u\right\|^2 .$$

Let $u \in C^\infty(M)$ and $f = u\circ\pi \in C^\infty(\mathfrak{M})$. A straightforward calculation shows that

$$(D^2 f)(X^\uparrow, Y^\uparrow) = (\nabla^2 u)(X,Y) - \Omega(X,Y)u_0\,, \tag{65}$$

$$(D^2 f)(X^\uparrow, T^\uparrow) = (\nabla^2 u)(T,X) - (\phi X)(u), \tag{66}$$

$$(D^2 f)(X^\uparrow, S) = -(1/2)\,(JX)(u), \tag{67}$$

$$(D^2 f)(T^\uparrow, T^\uparrow) = T(u_0) - 2V(u), \tag{68}$$

$$(D^2 f)(T^\uparrow, S) = 0, \tag{69}$$

$$(D^2 f)(S,S) = 0, \tag{70}$$

for every $X, Y \in H(M)$. Consequently

$$F_\theta^*(D^2 f\,,\,D^2 f) = \left\|\Pi_H \nabla^2 u\right\|^2 + 2nu_0^2 - 2\,\mathrm{div}\left(J\nabla^H u\right)(u_0) + \tag{71}$$

$$+4\left\{(J\nabla^H u)(u_0) - \left(\tau J\nabla^H u + \phi J\nabla^H u\right)(u)\right\}.$$

By a result of J.M. Lee, [29], if $f = u \circ \pi$ then $\Box f = (\Delta_b u) \circ \pi$ hence

$$(Df)(\Box f) = (\nabla^H u)(\Delta_b u), \quad F_\theta(Df, Df) = \|\nabla^H u\|^2 . \tag{72}$$

Finally (by taking into account the identities (64), (71) and (72) the Bochner-Lichnerowicz formula (21) becomes

$$\begin{aligned} -\frac{1}{2}\,\Delta_b\left(\|\nabla^H u\|^2\right) &= \left\|\Pi_H \nabla^2 u\right\|^2 + 4nu_0^2 - 2\,\mathrm{div}\left(J\nabla^H u\right) u_0 + \\ &+4\left\{(J\nabla^H u)(u_0) - \left(\tau J\nabla^H u + \phi J\nabla^H u\right)(u)\right\} - (\nabla^H u)(\Delta_b u) + \\ &+\frac{n+1}{n+2}\left\{\mathrm{Ric}_\nabla(\nabla^H u, \nabla^H u) + 3\,A(\nabla^H u, J\nabla^H u)\right\} + \frac{\rho}{2(n+1)(n+2)}\,\|\nabla^H u\|^2 . \end{aligned} \tag{73}$$

The term $(\phi J\nabla^H u)(u)$ may be expressed in terms of pseudohermitian Ricci curvature and torsion. As $J\nabla^H u = i\left(u^\alpha T_\alpha - u^{\overline{\alpha}} T_{\overline{\alpha}}\right)$ with $u^\alpha = g^{\alpha\overline{\beta}} u_{\overline{\beta}}$ and $u_{\overline{\beta}} = T_{\overline{\beta}}(u)$ one has (by (29))

$$\begin{aligned} \phi J\nabla^H u &= i\left(u^\alpha \phi_\alpha{}^\beta T_\beta - u^{\overline{\alpha}} \phi_{\overline{\alpha}}{}^{\overline{\beta}} T_{\overline{\beta}}\right) = \\ &= -\frac{1}{2(n+2)}\left\{g^{\beta\overline{\nu}}\mathrm{Ric}_\nabla\left(\left(\nabla^H u\right)^{1,0}, T_{\overline{\nu}}\right) T_\beta + \right. \\ &\left. + g^{\overline{\beta}\nu}\mathrm{Ric}_\nabla\left(\left(\nabla^H u\right)^{0,1}, T_\nu\right) T_{\overline{\beta}}\right\} + \frac{\rho}{4(n+1)(n+2)}\,\nabla^H u \end{aligned}$$

hence (as Ric_∇ is symmetric on $H(M) \otimes H(M)$)

$$\begin{aligned} (\phi J\nabla^H u)(u) &= \frac{\rho}{4(n+1)(n+2)}\,\|\nabla^H u\|^2 - \\ &-\frac{1}{n+2}\,\mathrm{Ric}_\nabla\left(\left(\nabla^H u\right)^{1,0}, \left(\nabla^H u\right)^{0,1}\right). \end{aligned} \tag{74}$$

Formula (32) implies

$$\mathrm{Ric}_\nabla\left(X^{1,0}, X^{0,1}\right) = \frac{1}{2}\left\{\mathrm{Ric}_\nabla(X, X) - (n-1)\,A(X, JX)\right\} \tag{75}$$

for any $X \in H(M)$. Hence (by (75) with $X = \nabla^H u$) formula (74) becomes

$$\begin{aligned} (\phi J\nabla^H u)(u) &= \frac{\rho}{4(n+1)(n+2)}\,\|\nabla^H u\|^2 - \\ &-\frac{1}{2(n+2)}\left\{\mathrm{Ric}_\nabla\left(\nabla^H u, \nabla^H u\right) - (n-1)\,A\left(\nabla^H u, J\nabla^H u\right)\right\}. \end{aligned} \tag{76}$$

Let us substitute from (76) and $(\tau J\nabla^H u)(u) = A(\nabla^H u, J\nabla^H u)$ into (73). We obtain

$$\begin{aligned} -\frac{1}{2}\,\Delta_b\left(\|\nabla^H u\|^2\right) &= \left\|\Pi_H \nabla^2 u\right\|^2 - (\nabla^H u)(\Delta_b u) + 4nu_0^2 + \\ &+4(J\nabla^H u)(u_0) - 2\,\mathrm{div}(J\nabla^H u)\,u_0 + \frac{n+3}{n+2}\,\mathrm{Ric}_\nabla\left(\nabla^H u, \nabla^H u\right) - \end{aligned} \tag{77}$$

$$-\frac{\rho}{2(n+1)(n+2)}\,\|\nabla^H u\|^2 - \frac{3(n+1)}{n+2}\,A(\nabla^H u\,,\,J\nabla^H u).$$

A straightforward calculation shows that for any $u \in C^\infty(M)$

$$\operatorname{div}\left(J\nabla^H u\right) = 2nu_0\,. \tag{78}$$

By (78) identity (77) simplifies to

$$-\frac{1}{2}\,\Delta_b\left(\|\nabla^H u\|^2\right) = \left\|\Pi_H \nabla^2 u\right\|^2 - (\nabla^H u)(\Delta_b u) + \tag{79}$$

$$+4(J\nabla^H u)(u_0) + \frac{n+3}{n+2}\,\mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right) -$$

$$-\frac{\rho}{2(n+1)(n+2)}\,\|\nabla^H u\|^2 - \frac{3(n+1)}{n+2}\,A(\nabla^H u\,,\,J\nabla^H u).$$

(the *pseudohermitian Bochner-Lichnerowicz* formula). Let us integrate over M and observe that (by Green's lemma and (78))

$$\int_M (J\nabla^H u)(u_0)\,\Psi_\theta = -\int_M u_0 \operatorname{div}(J\nabla^H u)\,\Psi_\theta = -2n\,\|u_0\|^2_{L^2}\ .$$

We obtain

$$\left\|\Pi_H \nabla^2 u\right\|^2_{L^2} - 8n\ \|u_0\|^2_{L^2} + \tag{80}$$

$$+\int_M \left\{\frac{n+3}{n+2}\,\mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right) - \frac{3(n+1)}{n+2}\,A\left(\nabla^H u\,,\,J\nabla^H u\right)\right\}\,\Psi_\theta =$$

$$= \int_M \left(\nabla^H u\right)(\Delta_b u)\ \Psi_\theta + \frac{1}{2(n+1)(n+2)}\int_M \rho\ \left\|\nabla^H u\right\|^2\ \Psi_\theta$$

(the *integral* pseudohermitian Bochner-Lichnerowicz formula).

6. A lower bound on $\lambda_1(\theta)$

Let $\lambda \in \sigma(\Delta_b)$ be an eigenvalue of Δ_b and $u \in \mathrm{Eigen}(\Delta_b\,,\,\lambda)$ an eigenfunction corresponding to λ. With these data

$$\int_M \left(\nabla^H u\right)(\Delta_b u)\ \Psi_\theta = \lambda\ \left\|\nabla^H u\right\|^2_{L^2}\ . \tag{81}$$

On the other hand (cf. (27) in [4], p. 88)

$$\left\|\Pi_H \nabla^2 u\right\|^2 \geq \frac{1}{2n}\ (\Delta_b u)^2 \tag{82}$$

everywhere on M. Moreover (by Green's lemma)

$$\|\Delta_b u\|^2_{L^2} = \lambda \int_M u\,\Delta_b u\ \Psi_\theta = \lambda\,\|\nabla^H u\|^2_{L^2}\,. \tag{83}$$

By our assumption (5)

$$\int_M \mathrm{Ric}_\nabla\left(\nabla^H u\,,\,\nabla^H u\right)\ \Psi_\theta \geq k\ \left\|\nabla^H u\right\|^2_{L^2}\ . \tag{84}$$

Moreover (by (5) with $X = E_a$)

$$\rho \geq nk. \tag{85}$$

In particular $\rho_0 \equiv \sup_{x\in M} \rho(x) > 0$ and

$$\int_M \rho \|\nabla^H u\|^2 \, \Psi_\theta \leq \rho_0 \, \left\|\nabla^H u\right\|^2_{L^2} . \tag{86}$$

For any $X, Y \in H(M)$ (by Cauchy-Schwartz inequality)

$$|A(X,Y)| = |G_\theta(X, \tau Y)| \leq \|X\| \, \|\tau Y\| \leq \|\tau\| \, \|X\| \, \|Y\|,$$

$$\|\tau\|_x = \sup\left\{G_{\theta,x}\left(\tau_x v \, , \, \tau_x v\right) : v \in H(M)_x \, , \; \; G_{\theta,x}(v,v) = 1\right\}, \quad x \in M.$$

Consequently

$$\int_M A\left(\nabla^H u \, , \, J\nabla^H u\right) \leq \tau_0 \left\|\nabla^H u\right\|^2_{L^2} \tag{87}$$

where $\tau_0 = \sup_{x\in M} \|\tau\|_x$. The integral Bochner-Lichnerowicz formula (80) reads (by (81))

$$0 = \left\|\Pi_H \nabla^2 u\right\|^2_{L^2} - 8n \, \|u_0\|^2_{L^2} +$$

$$+ \int_M \left\{ \frac{n+3}{n+2} \, \mathrm{Ric}_\nabla \left(\nabla^H u \, , \, \nabla^H u\right) - \frac{3(n+1)}{n+2} \, A\left(\nabla^H u \, , \, J\nabla^H u\right) \right\} \, \Psi_\theta -$$

$$-\lambda \, \left\|\nabla^H u\right\|^2_{L^2} - \frac{1}{2(n+1)(n+2)} \int_M \rho \, \left\|\nabla^H u\right\|^2 \, \Psi_\theta \geq$$

(by (82) and (84)-(87))

$$\geq \frac{1}{2n} \, \|\Delta_b u\|^2_{L^2} - 8n \, \|u_0\|^2_{L^2} + \left[\frac{(n+3)k}{n+2} - \frac{3(n+1)\tau_0}{n+2} \right] \left\|\nabla^H u\right\|^2_{L^2} -$$

$$-\lambda \, \left\|\nabla^H u\right\|^2_{L^2} - \frac{\rho_0}{2(n+1)(n+2)} \, \left\|\nabla^H u\right\|^2_{L^2}$$

so that (by (83))

$$\left\{ \frac{1}{2n} - 1 + \frac{1}{\lambda} \left[\frac{(n+3)k}{n+2} - \frac{3(n+1)\tau_0}{n+2} - \right.\right.$$

$$\left.\left. - \frac{\rho_0}{2(n+1)(n+2)} \right] \right\} \|\Delta_b u\|^2_{L^2} \leq 8n \, \|u_0\|^2_{L^2} \, .$$

Finally (by (83) and Chang-Chiu inequality (91) in Appendix A)

$$-\frac{2n+3}{n+2} + \frac{1}{\lambda} \left\{ \frac{(n+3)k}{2(n+1)} - \frac{(11n+19)\tau_0}{n+2} - \frac{\rho_0}{2(n+1)(n+2)} \right\} \leq 0$$

or

$$\lambda \geq \frac{2n}{(n+2)(n+3)} \left\{ (n+3)k - (11n+19)\tau_0 - \frac{\rho_0}{2(n+1)} \right\} \tag{88}$$

which is the announced lower bound on $\lambda_1(\theta)$ (cf. (6) above). Of course this is useful only when

$$k > \frac{(11n+19)\tau_0}{n+3} + \frac{\rho_0}{2(n+1)(n+3)} . \tag{89}$$

In particular (by (85)) it must be $k > 2(n+1)(11n+19)\tau_0/[(n+2)(2n+3)]$. To parallel the estimate (6) in Theorem 1 to A. Greenleaf's estimate (7) let g_θ be a Sasakian metric. If this is the case the assumption

$$R_{\lambda\overline{\mu}} Z^\lambda Z^{\overline{\mu}} + \frac{in}{2}\left(A_{\overline{\alpha}\overline{\beta}} Z^{\overline{\alpha}} Z^{\overline{\beta}} - A_{\alpha\beta} Z^\alpha Z^\beta\right) \geq k\, g_{\lambda\overline{\mu}} Z^\lambda Z^{\overline{\mu}}$$

(with $Z^{\overline{\alpha}} = \overline{Z^\alpha}$) in [23], p. 192, is equivalent to (5). Also (6) becomes

$$\lambda_1(\theta) \geq \frac{2n}{(n+2)(n+3)}\left\{(n+3)k - \frac{\rho_0}{2(n+1)}\right\} \tag{90}$$

and right hand side of (90) is larger (closer to $\lambda_1(\theta)$ from below) than right hand side of (7) precisely when (8) holds. In particular if $M = S^{2n+1}$ then the Tanaka-Webster connection of the canonical contact form $\theta = (i/2)\left(\overline{\partial} - \partial\right)|z|^2$ has curvature (cf. [12])

$$R^\nabla(X,Y)Z = g_\theta(Y,Z)X - g_\theta(X,Z)Y + \Omega(X,Z)JY - \Omega(Y,Z)JX + 2\Omega(X,Y)JZ$$

for any $X, Y, Z \in H(S^{2n+1})$. Consequently the pseudohermitian Ricci and scalar curvature of the sphere are $R_{\lambda\overline{\mu}} = 2(n+1)\, g_{\lambda\overline{\mu}}$ and $\rho = 2n(n+1)$ so that (8) becomes $k > 2(n+1)/(n+3)$ (which is clearly satisfied by $k = 2(n+1)$).

Appendix A. The Chang-Chiu inequality

The purpose of Appendix A is to give a proof of

$$4n\, \|u_0\|^2_{L^2} \leq \frac{1}{n}\, \|\Delta_b u\|^2_{L^2} + 4\,\tau_0\, \left\|\nabla^H u\right\|^2_{L^2} \tag{91}$$

for any $u \in C^\infty(M, \mathbb{R})$ (compare[2] to (3.5) in [10], p. 270). This is referred to as the *Chang-Chiu inequality*. To prove (91) let us contract (13) by u^β so that to obtain $u^\beta \nabla_0 u_\beta = u^\beta \nabla_\beta u_0 - A_{\alpha\beta} u^\alpha u^\beta$ or

$$u^\beta\, \nabla_0 u_\beta = \nabla_\beta \left(u_0 u^\beta\right) - u_0\, \nabla_\beta u^\beta - A_{\alpha\beta} u^\alpha u^\beta . \tag{92}$$

[2] Discrepancies among (91) and (3.5) in [10], p. 270, are due to the different convention as to wedge products of 1-forms producing the additional 2 factor in (12). Cf. also (1.62) in [12], p. 39, and (9.7) in [12], p. 424. Through this paper conventions as to wedge products and exterior differentiation calculus are those in [28], p. 35-36.

On the other hand (by (12)) $\nabla_\beta u^\beta = \nabla_{\overline{\beta}} u^{\overline{\beta}} - 2in\, u_0$ so that (by substitution into (92))

$$u^\beta \nabla_0 u_\beta + u_0\, \nabla_{\overline{\beta}} u^{\overline{\beta}} = 2in\, u_0^2 - A_{\alpha\beta} u^\alpha u^\beta + \nabla_\beta \left(u_0 u^\beta\right). \tag{93}$$

Next (again by (13)) $u_0\, \nabla_{\overline{\beta}} u^{\overline{\beta}} = \nabla_{\overline{\beta}} \left(u_0 u^{\overline{\beta}}\right) - u^{\overline{\beta}} \left(\nabla_0 u_{\overline{\beta}} + u_\gamma A^\gamma_{\overline{\beta}}\right)$ hence (by substitution of $u_0\, \nabla_{\overline{\beta}} u^{\overline{\beta}}$ into (93))

$$i\left(u^{\overline{\beta}}\, \nabla_0 u_{\overline{\beta}} - u^\beta\, \nabla_0 u_\beta\right) = \tag{94}$$

$$= 2nu_0^2 + i\left(A_{\alpha\beta} u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}}\right) + i\left\{\nabla_{\overline{\alpha}} \left(u_0 u^{\overline{\alpha}}\right) - \nabla_\alpha \left(u_0 u^\alpha\right)\right\}$$

(compare to (2.4) in Lemma 2.2, [10], p. 268). Calculations are performed with respect to an arbitrary local frame $\{T_\alpha : 1 \le \alpha \le n\}$ in $T_{1,0}(M)$ (rather than a G_θ-orthonormal frame, as in [10]). The next step is to evaluate the left hand side of (94) in terms of the operator $P + \overline{P}$. One has $u_0 = (i/2n) \left(\nabla_\beta u^\beta - \nabla_{\overline{\beta}} u^{\overline{\beta}}\right)$ hence (by (13))

$$u^{\overline{\alpha}} \nabla_0 u_{\overline{\alpha}} = \frac{i}{2n}\, u^{\overline{\alpha}} \left(u_{\overline{\alpha}}{}^{\overline{\gamma}}{}_{\overline{\gamma}} - u_{\overline{\alpha}}{}^{\gamma}{}_{\gamma}\right) - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}}\,. \tag{95}$$

Using $P_{\overline{\alpha}} u \equiv u_{\overline{\alpha}}{}^{\gamma}{}_{\gamma} - 2ni\, A_{\overline{\alpha}\overline{\beta}} u^{\overline{\beta}}$ the identity (95) becomes

$$i\, u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} = \frac{1}{2n}\, u^{\overline{\alpha}} \left(P_{\overline{\alpha}} u - u_{\overline{\alpha}}{}^{\overline{\gamma}}{}_{\overline{\gamma}}\right). \tag{96}$$

Let us take the complex conjugate of (96) and add the resulting equation to (96). We obtain

$$2ni\, \left(u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} - u^\beta\, \nabla_0 u_\beta\right) = u^{\overline{\alpha}} P_{\overline{\alpha}} u + u^\alpha P_\alpha u - \left\{u^{\overline{\alpha}}\, u_{\overline{\alpha}}{}^{\overline{\gamma}}{}_{\overline{\gamma}} + u^\alpha\, u_\alpha{}^{\gamma}{}_{\gamma}\right\} \tag{97}$$

where $P_\alpha u \equiv u_\alpha{}^{\overline{\gamma}}{}_{\overline{\gamma}} + 2ni A_{\alpha\beta} u^\beta$. Let us replace $u^\alpha\; u_\alpha{}^\beta{}_\beta + u^{\overline{\alpha}}\; u_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\beta}}$ from (18) into (97). We obtain

$$2ni\left(u^{\overline{\alpha}}\, \nabla_0 u_{\overline{\alpha}} - u^\alpha\, \nabla_0 u_\alpha\right) = 2\left(u^\alpha\, P_\alpha u + u^{\overline{\alpha}} P_{\overline{\alpha}} u\right) - \tag{98}$$

$$-2ni\left(A_{\alpha\beta} u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} u^{\overline{\beta}}\right) + \left(\nabla^H u\right)(\Delta_b u).$$

Finally substitution from (98) into (94) leads to

$$2\left(u^\alpha\, P_\alpha + u^{\overline{\alpha}}\, P_{\overline{\alpha}} u\right) + \left(\nabla^H u\right)(\Delta_b u) = \tag{99}$$

$$= 4n^2 u_0^2 + 4ni\left(A_{\alpha\beta} u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}\overline{\beta}}\right) + 2ni\left\{\nabla_{\overline{\alpha}} \left(u_0 u^{\overline{\alpha}}\right) - \nabla_\alpha \left(u_0 u^\alpha\right)\right\}.$$

Let us observe that

$$i\left(A_{\alpha\beta} u^\alpha u^\beta - A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}\overline{\beta}}\right) = A\left(\nabla^H u\,,\, J\nabla^H u\right),$$

$$i\left\{\nabla_\alpha \left(u_0 u^\alpha\right) - \nabla_{\overline{\alpha}} \left(u_0 u^{\overline{\alpha}}\right)\right\} = \operatorname{div}\left(u_0\, J\nabla^H u\right),$$

and $u^\alpha\, P_\alpha + u^{\overline{\alpha}}\, P_{\overline{\alpha}} u = g^*_\theta(Lu\,,\, d_b u)$ where $L = P + \overline{P}$. Then (99) becomes

$$2\, g^*_\theta\, (Lu\,,\, d_b u) + \left(\nabla^H u\right) (\Delta_b u) = 4n^2\, u_0^2 + \tag{100}$$

$$+4n\, A\left(\nabla^H u\,,\, J\nabla^H u\right) - 2n\, \mathrm{div}\left(u_0\, J\nabla^H u\right).$$

Let us integrate over M and use Green's lemma. Then (by Lemma 1)

$$-2\int_M (P_0 u) u\, \Psi_\theta + \int_M \left(\nabla^H u\right) (\Delta_b u)\, \Psi_\theta = \tag{101}$$

$$= 4n^2\, \|u_0\|^2_{L^2} + 4n \int_M A\left(\nabla^H u\,,\, J\nabla^H u\right)\, \Psi_\theta\,.$$

Also (again by Green's lemma)

$$\int_M \left(\nabla^H u\right) (\Delta_b u)\, \Psi_\theta = \int_M (\Delta_b u)^2\;\; \Psi_\theta = \|\Delta_b u\|^2_{L^2}\;.$$

Finally as P_0 is nonnegative (87) and (101) lead to (91). Q.e.d.

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